Probability, Random Variables, and Random Signal Principles

4^{TH} EDITION

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CHAPTER - 1

Probability

1·3-1 A die is tossed. Find the probabilities of the events $A = \{$odd number shows up$\}$, $B = \{$number larger than 3 shows up$\}$, $A \cup B$, and $A \cap B$.

Sol: The events are $A = \{1, 3, 5\}$, $B = \{4, 5, 6\}$, $A \cup B = \{1, 3, 4, 5, 6\}$, $A \cap B = \{5\}$

* By assuming a fair die, the probability of each of the six mutually exclusive outcomes is $1/6$

Thus from axiom 3

$$P(A) = 1/6 + 1/6 + 1/6 = 3/6 = 1/2$$

$P(B) = 1/2$, $P(A \cup B) = 5/6$ and $P(A \cap B) = 1/6$.

1·3-2 In a game of dice, a "shooter" can win outright if the sum of two numbers showing up is either 7 or 11 when two dice are thrown. What is the probability of winning outright?

Sol: There are 36 possible mutually exclusive outcomes, each with probability $1/36$. Only six produces a sum 7 : (1,6), (2,5), (3,4), (4,3), (5,2), (6,1). Two produces a sum 11 : (5,6) (6,5).

Thus, eight outcomes satisfy the event 7 or 11. Therefore probability of winning, i.e., $P(\text{sum is } 7 \text{ or } 11) = 8/36 = 2/9$.

1·3-3 A pointer is spun on a fair wheel of chance having its periphery labeled from 0 to 100.

a) What is the sample space for this experiment?

b) What is the probability that the pointer will stop between 20 and 35?

c) What is the probability that the wheel will stop on 58?

Sol: (a) Sample space $S = \{0 < s \leq 100\}$

(b) $P\{20 < s \leq 35\} = 35-20/100 = 15/100$

(c) $P\{s = 58\} = 0$, since the number 58 is only one of an infinite number of numbers in sample space S.

1·3-4 An experiment has a sample space with 10 equally likely elements $S = \{a_1, a_2, \ldots a_{10}\}$. Three events are defined as $A = \{a_1, a_5, a_9\}$, $B = \{a_1, a_2, a_6, a_9\}$ and

$C=\{a_6, a_9\}$. Find the Probabilities of (a) $A \cup C$ (b) $B \cup \bar{C}$ (c) $A \cap (B \cup C)$ (d) $\overline{A \cup B}$ (e) $(A \cup B) \cap C$

Sol: (a) $P(A \cup C) = P\{a_1, a_5, a_6, a_9\} = 4/10$

(b) $P(B \cup \bar{C}) = P\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\} = P(S) = 1$

(c) $P(A \cap (B \cup C)) = P\{a_1, a_9\} = 2/10$

(d) $P(\overline{A \cup B}) = P\{a_3, a_4, a_7, a_8, a_{10}\} = 5/10 = 1/2$

(e) $P(A \cup (B \cap C)) = P\{a_6, a_9\} = 2/10 = 1/2$.

1.3-5 Let A be an arbitary event. Show that $P(\bar{A}) = 1 - P(A)$.

Sol: A and $\bar{A}$ are mutually exclusive and span the entire sample space. Therefore, from axiom 3: $P(A \cup \bar{A}) = P(S) = 1$

$$P(A) + P(\bar{A}) = 1 \quad (\text{or}) \quad P(\bar{A}) = 1 - P(A).$$

1.3-6 An experiment consists of rolling a single die. Two events are defined as: $A = \{a\ 6\ \text{shows up}\}$ and $B = \{a\ 2\ \text{or}\ a\ 5\ \text{shows up}\}$.

(a) Find $P(A)$ and $P(B)$.

(b) Define a third event C so that $P(C) = 1 - P(A) - P(B)$

Sol: (a) $P(A) = P\{6\} = 1/6$; $P(B) = P\{2,5\} = 2/6 = 1/3$

(b) $P(A) + P(B) + P(C) = 1$ is generated if C is exclusive of A and B and comprises the balance of the sample space.

Thus $C = \overline{A \cup B} = \{1,3,4\}$

1.3-7 In a box there are 500 colored balls: 75 black, 150 green, 175 red, 70 white and 30 blue. What is the probability of selecting a ball of each color?.

Sol: $P(\text{black}) = 75/500$, $P(\text{green}) = 150/500$, $P(\text{red}) = 175/500$, $P(\text{white}) = 70/500$ and $P(\text{blue}) = 30/500$.

1.3-8 A single card is drawn from a 52-card deck.

(a) what is the probability that the card is a jack?

(b) what is the probability the card will be a 5 or smaller?

(c) what is the probability that the card is red 10?

Sol: (a) $P(\text{Jack}) = 4/52 = 1/13$.

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(b) P(5 or smaller) = P(4 fives + 4 fours + 4 threes + 4 twos) = $16/52 = 4/13$

(c) P(red 10) = $2/52$

1.3-9 A Pair of fair dice are thrown in a gambling problem. Person A wins if the sum of numbers showing up is six or less and one of the dice shows four. Person B wins if the sum is five or more and one of the dice shows four. Find (a) The Probability that A wins (b) the Probability of B winning and (c) the Probability that both A and B win.

Sol: (a) $P(A \text{ wins}) = P(2,4) + P(1,4) + P(4,1) + P(4,2) = 4/36$

(b) $P(B \text{ wins}) = P(4,1) + P(4,2) + P(4,3) + P(4,4) + P(4,5) + P(4,6) + P(1,4) + P(2,4) + P(3,4) + P(5,4) + P(6,4) = 11/36$

(c) $P(A \text{ and } B \text{ wins}) = P(A \text{ wins}) = 4/36$ $(\because A \subset B)$.

1.3-10 You (Person A) and two others (B and C) each toss a fair Coin in a two-step gambling game. In step 1 the Person whose toss is not match to either of the other two is "odd man out". Only the remaining two whose Coins match go on to step 2 to resolve the ultimate winner.

(a) what is the Probability you will advance to step 2 after the first toss?

(b) what is the Probability you will be out after the first toss?

(c) what is the Probability that no one will be out after the first toss?

Sol: (a) $P(\text{stay in after one toss}) = P(H,H,T) + P(H,T,H) + P(T,T,H) + P(T,H,T) = 4/8$.

(b) $P(\text{Out after first toss}) = P(H,T,T) + P(T,H,H) = 2/8$.

(c) P(no "odd man") $= P(H,H,H) + P(T,T,T) = 2/8$.

1.3-11 A Particular electronic device is known to Contain only 10Ω, 22Ω and 48Ω resistors, but these resistors may have 0.25W, 0.5W (or) 1-W ratings, depending on how purchases are made to minimize cost. Historically, it is found that the Probabilites of 10Ω resistors being 0.25W, 0.5W or 1W are 0.08, 0.10 and 0.01 respectively. For the 22Ω resistors the similar probabilities are 0.20, 0.26 and 0.05. It is also historically found that the probabilities are 0.40, 0.51 and 0.09 that any resistors are 0.25W, 0.50W and 1W, respectively. what are the probabilities

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that the 48 Ω resistors are (a) 0.25 W (b) 0.50 W and (c) 1 W ?

	0.25 W	0.50 W	1.0 W
10 Ω	0.08	0.10	0.01
22 Ω	0.20	0.26	0.05
48 Ω	0.12	0.15	0.03
	0.40	0.51	0.09

From table :

(a) $P(48\,\Omega \text{ and } 0.25\,W) = 0.12$ (b) $P(48\,\Omega \text{ and } 0.5\,W) = 0.15$

(c) $P(48\,\Omega \text{ and } 1.0\,W) = 0.03$.

1.3-12: For the sample space defined in 1.3-2, find the probabilities that
(a) One die will show a 2 and the other will show a 3 or larger, and
(b) the sum of the two numbers showing up will be 4 or less or will be 10 or more.

Sol: (a) $P(\text{One is a 2 and other is 3 or more}) = P(2,3) + P(2,4) + P(2,5) + P(2,6) + P(3,2) + P(4,2) + P(5,2) + P(6,2) = 8/36 = 2/9$.

(b) $P(10 \le \text{sum and sum} \le 4) = P(4,6) + P(5,5) + P(5,6) + P(6,4) + P(6,5) + P(6,6) + P(1,1) + P(1,2) + P(1,3) + P(2,1) + P(2,2) + P(3,1) = 12/36 = 1/3$

1.3-13 In a game two dice are thrown. Let one die be "weighted" so that a 4 shows up with probability 2/7, while its other numbers all have probabilities 1/7. The same probabilities apply to the other die except the number 3 is "weighted". Determine the probability the shooter will win outright by having the sum of the numbers showing up be 7. What would be the probability for fair dice?

Sol: $P\{7\} = P(6,1) + P(5,2) + P(4,3) + P(3,4) + P(2,5) + P(1,6) = 9/49 \cong 0.1837$

For fair dice each of outcome has probability 1/36. So

$P\{7\} = 6/36 = 1/6 \cong 0.166$.

$\text{Improvement} = (9/49)/(6/36) \cong 1.102$ (10.2%)

Example 1.4-1 : In a box there are 100 resistors having resistance and tolerance as shown. Let a resistor be selected from the box and assumes each resistor has the same likelihood of being choosen. Define 3 events : A as "draw a 47 Ω resistor", B as "draw a resistor with 5% tolerance" and C as "draw a 100 Ω resistor". Find all applicable probabilities.

1.4-1 Two cards are drawn from a 52-card deck (the first is not replaced)

(a) Give the first card is queen, what is the Probability that the second is also queen.

(b) Repeat Part (a) for the first card a queen and second card a 7.

(c) what is the Probability that both cards will be a queen.

Sol: (a) $P(\text{Second is queen} / \text{first is queen}) = 3/51$

(b) $P(\text{Second is seven} / \text{first is queen}) = 4/51$

(c) $P(\text{both are queen}) = P(\text{queen} / \text{queen}) \cdot P(\text{queen}) = \frac{3}{51} \cdot 4/52$

1.4-2 An ordinary 52-card deck is throughly shuffled. You are dealt four cards up. what is the Probability that all four cards are sevens?

Sol: $P(\text{Four Seven}) = P(\text{fourth is seven} / \text{first three sevens}) \cdot P(\text{first three sevens})$

$= 1/49 \cdot P(\text{first three sevens})$

$= 1/49 \cdot P(\text{third is seven} / \text{first two sevens}) \cdot P(\text{first two sevens})$

$= 1/49 \cdot 2/50 \cdot P(\text{first two sevens})$

$= 1/49 \cdot 2/50 \cdot P(\text{Second is seven} / \text{first is seven}) \cdot P(\text{first is seven})$

$= 1/49 \cdot 2/50 \cdot 3/51 \cdot 4/52$

$= 1/270725 = 3.694 \times 10^{-6}$

Resistance	5%	10%	Total
22	10	14	24
47	28	16	44
100	24	8	32
	62	38	100

1.4-3 For the resistor selection experiment of Example 1.4-1, define event D as "draw a 22 Ω resistor", E as "draw a resistor with 10% tolerance". Find $P(D)$, $P(E)$, $P(D \cap E)$, $P(D/E)$ and $P(E/D)$.

Sol: $P(D) = 24/100$, $P(E) = 38/100$, $P(D \cap E) = 14/100$, $P(D/E) = 14/38$

$P(E/D) = 14/24$.

1.4-4 For the resistor selection experiment of Example 1.4-1, define two mutually exclusive events B_1 and B_2 such that $B_1 \cup B_2 = S$.

(a) Use the total Probability theorem to find the Probability of the event "Select a 22-Ω resistor", denoted D.

(b) Use Bayes theorem to find the probability that the resistor selected had 5% tolerance, given it was 22 Ω.

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Sol: Define B_1 = "draw a 5% tolerance resistor"
B_2 = "draw a 10% tolerance resistor".

(a) $P(D) = P(D/B_1)\,P(B_1) + P(D/B_2)\cdot P(B_2)$
$= 10/62 \cdot 62/100 + 14/38 \cdot 38/100 = 24/100$

(b) $P(5\%/22\Omega) = P(B_1/D) = P(D/B_1)\cdot P(B_1) / P(D)$
$= 100/62 \cdot 62/100 \,/\, 24/100 = 10/24$.

1.4-5 In three boxes there are capacitors as shown in table.

Value (μF)	Number in box 1	2	3	Totals
0.01 μF	20	95	25	140
0.1	55	35	75	165
1.0	70	80	145	295
Totals	145	210	245	600

An experiment Consists of first randomly selecting a box, assuming each has the same likelihood of selection, and then selecting a capacitor from the choosen box.

(a) what is the Probability of selecting a 0.01 μF capacitor, given that box 2 is selected?

(b) If a 0.01 μF Capacitor is selected, what is the Probability it came from box 3? (Hint: Use Bayes and Total Probability theorem).

Sol: (a) $P(0.01\,\mu F / \text{box } 2) = 95/210$

(b) $P(\text{box } 3 / 0.01\,\mu F) = ?$

$P(0.01\,\mu F/\text{box}3)\cdot P(\text{box}3) = P(\text{box}3/0.01\,\mu F)\cdot P(0.01\,\mu F)$

Thus

$$P(\text{box}3/0.01\,\mu F) = P(0.01\,\mu F/\text{box}3)\cdot P(\text{box}3) \,/\, P(0.01\,\mu F)$$

From the total probability theorem:

$P(0.01\,\mu F) = P(0.01\,\mu F/\text{box}1)\cdot P(\text{box}1) + P(0.01\,\mu F/\text{box}2)\,P(\text{box}2)$
$+ P(0.01\,\mu F/\text{box}3)\cdot P(\text{Box}3)$
$= 20/145 \cdot 1/3 + 95/210 \cdot 1/3 + 25/245 \cdot 1/3$

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Thus

$$P(\text{box}3/0.01\,\mu F) = 870/5903 \cong 0.1474.$$

1.4-6 For Problem 1.4-5, list the nine Conditional Probabilities of Capacitor selection, given Certain box selections.

Sol: $P(0.01\,\mu F/\text{box}1) = 20/145$; $P(0.01\,\mu F/\text{box}2) = 95/210$

$P(0.01\,\mu F/\text{box}3) = 25/245$

$P(0.1\,\mu F/\text{box}1) = 55/145$; $P(0.1\,\mu F/\text{box}2) = 35/210$; $P(0.1\,\mu F/\text{box}3) = 75/245$.

$P(1.0\,\mu F/\text{box}1) = 70/145$; $P(1.0\,\mu F/\text{box}2) = 80/210$; $P(1.0\,\mu F/\text{box}3) = 145/245$.

1.4-7 Rework example 1.4-2 if $P(B_1) = 0.6$, $P(B_2) = 0.4$, $P(A_1/B_1) = P(A_2/B_2) = 0.95$, and $P(A_2/B_1) = P(A_1/B_2) = 0.05$.

Sol: $P(A_1) = P(A_1/B_1)\cdot P(B_1) + P(A_1/B_2)\cdot P(B_2)$

$= 0.95\,(0.6) + 0.05\,(0.4) = 0.59$

$P(A_2) = P(A_2/B_1)\,P(B_1) + P(A_2/B_2)\cdot P(B_2) = 0.05(0.6) + 0.95\,(0.4) = 0.41$

$P(B_1/A_1) = P(A_1/B_1)\cdot P(B_1) / P(A_1) = 0.95\,(0.6)/0.59 = 0.966$

$P(B_2/A_2) = P(A_2/B_2)\cdot P(B_2) / P(A_2) = 0.95\,(0.4)/0.41 = 0.927$

$P(B_1/A_2) = P(A_2/B_1)\cdot P(B_1) / P(A_2) = 0.05\,(0.6)/0.41 = 0.073$

$P(B_2/A_1) = P(A_1/B_2)\cdot P(B_2) / P(A_1) = 0.05\,(0.4)/0.59 = 0.034.$

1.4-8 Rework Example 1.4-2 if $P(B_1) = 0.7$, $P(B_2) = 0.3$, $P(A_1/B_1) = P(A_2/B_2) = 1.0$ and $P(A_2/B_1) = P(A_1/B_2) = 0$. What type of channel does this System have?

Sol: $P(A_1) = 1.0\,(0.7) + 0.0\,(0.3) = 0.7$

$P(A_2) = 0.0\,(0.7) + 1.0\,(0.3) = 0.3$

$P(B_1/A_1) = 1.0\,(0.7)/0.7 = 1.0$; $P(B_2/A_2) = 1.0\,(0.3)/0.3 = 1.0$

$P(B_1/A_2) = 0$, $P(B_2/A_1) = 0$.

Thus system is ideal (noise-free) Since the probabilities of an error in symbol representation are zero.

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1·4-9 A Company Sells high fidelity amplifiers Capable of generating 10W, 25W and 50W of audio power. It has on hand 100 of the 10W units, of which 15% are defective, 70 of the 25W units with 10% defective, and 30 of the 50W units with 10% defective.

(a) what is the Probability that an amplifier Sold from the 10W units is defective?

(b) If each wattage amplifier Sells with equal likelihood, what is the probability of a randomly selected unit being 50W and defective?

(c) what is the Probability that a unit randomly selected for Sale is defective?

Sol:

Event	10W	25W	50W	Totals
D(Defective)	15	7	3	25
G(Good)	85	63	27	175
Totals	100	70	30	200

(a) $P(D/10W) = 15/100$

(b) $P(D \cap 50W) = P(D/50W) \cdot P(50W) = 3/30 \cdot 1/3 = 1/30$.

(c) $P(D) = P(D/10W) \cdot P(10W) + P(D/25W) \cdot P(25W) + P(D/50W) \cdot P(50W)$

$= 15/100 \cdot 1/3 + 7/70 \cdot 1/3 + 3/30 \cdot 1/3 = 0.35/3 \simeq 0.1167$.

1·4-10 A missile can be accidentally launched if two relays A and B both have failed. The Probabilities of A and B failing are known to be 0.01 and 0.03, respectively. It is also known that B is more likely to fail (Probability 0.06) if A has failed.

(a) what is the Probability of an accidental missile launch?

(b) what is the Probability That A will fail if B has failed.

(c) Are the events "A fails" and "B fails" statistically independent?

Sol: (a) $P(\text{launch}) = P(A \text{ fails} \cap B \text{ fails})$

$= P(B \text{ fails} / A \text{ fails}) \cdot P(A \text{ fails})$

$= 0.06\,(0.01) = 0.0006$

(b) $P(A \text{ fails} / B \text{ fails}) = P(A \text{ fail} \cap B \text{ fails}) / P(B \text{ fails}) = 6(10^{-4}) / 3(10^{-2}) = 0.02$

(c) $P(A \text{ fails}) \cdot P(B \text{ fails}) = 0.01(0.03) = 3 \times 10^{-4} \neq 6 \times 10^{-4}$

$$P(A \text{ fail} \cap B \text{ fail}) \neq P(A \text{ fail}) \cdot P(B \text{ fail})$$

Thus events "A fails" and "B fails" are not independent.

1·4-11 The Communication System of Example 1·4-2 is to be extended to the case of three transmitted symbols 0, 1 and 2. Define appropriate events A_i and B_i, $i = 1, 2, 3$, to represent symbols after and before the channel, respectively. Assume channel transmission probabilities are all equal at $P(A_i/B_j) = 0.1$, $i \neq j$, and are $P(A_i/B_j) = 0.8$ for $i = j = 1, 2, 3$ while symbol transmission probabilities are $P(B_1) = 0.5$, $P(B_2) = 0.3$ and $P(B_3) = 0.2$.

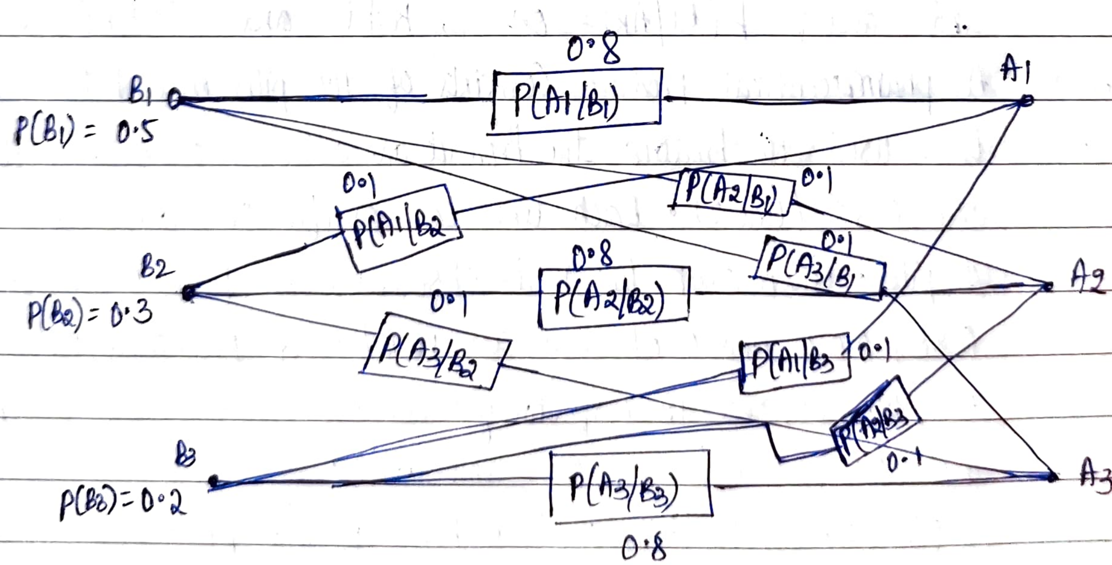
(a) Sketch the diagram analogous to fig. 1·4-2

(b) Compute received symbol probabilities $P(A_1)$, $P(A_2)$ and $P(A_3)$.

(c) Compute the a posteriori probabilities for this system.

(d) Repeat parts (b) and (c) for all transmission symbol probabilities equal. Note the effect.

Sol: (a)



(b) $P(A_1) = P(A_1/B_1) \cdot P(B_1) + P(A_1/B_2) \cdot P(B_2) + P(A_1/B_3) \cdot P(B_3)$

$= 0.8(0.5) + 0.1(0.3) + 0.1(0.2) = 0.45$

$P(A_2) = P(A_2/B_1) \cdot P(B_1) + P(A_2/B_2)\, P(B_2) + P(A_2/B_3) \cdot P(B_3)$

$= 0.1(0.5) + 0.8(0.3) + 0.1(0.2) = 0.31$

$P(A_3) = P(A_3/B_1) \cdot P(B_1) + P(A_3/B_2) \cdot P(B_2) + P(A_3/B_3) \cdot P(B_3)$

$= 0.1(0.5) + 0.1(0.3) + 0.8(0.2) = 0.24$

(c) $P(B_1/A_1) = 0.8(0.5)/0.45 = 0.8889 = P(A_1/B_1) \cdot P(B_1)/P(A_1)$ (Bayes rule)

$P(B_1/A_2) = 0.1(0.5)/0.31 = 0.1613 = P(A_2/B_1) \cdot P(B_1)/P(A_2)$

$P(B_1/A_3) = 0.1(0.5)/0.24 = 0.2083 = P(A_3/B_1) \cdot P(B_1)/P(A_3)$

$P(B_2/A_1) = 0.1(0.3)/0.45 = 0.0667 = P(A_1/B_2) \cdot P(B_2)/P(A_1)$

$P(B_2/A_2) = 0.8(0.3)/0.31 = 0.7742 = P(A_2/B_2) \cdot P(B_2)/P(A_2)$

$P(B_2/A_3) = 0.1(0.3)/0.24 = 0.1250 = P(A_3/B_2) \cdot P(B_2)/P(A_3)$

$P(B_3/A_1) = 0.1(0.2)/0.45 = 0.0444 = P(A_1/B_3) \cdot P(B_3)/P(A_1)$

$P(B_3/A_2) = 0.1(0.2)/0.31 = 0.0645 = P(A_2/B_3) \cdot P(B_3)/P(A_2)$

$P(B_3/A_3) = 0.8(0.2)/0.24 = 0.6667 = P(A_3/B_3) \cdot P(B_3)/P(A_3)$

(d) when $P(B_i) = 1/3$ $i = 1,2,3$ then $P(A_1) = 1/3[P(A_1/B_1) + P(A_1/B_2) + P(A_1/B_3)]$

$= 1/3[0.8 + 0.1 + 0.1] = 1/3$

lly $P(A_2) = P(A_3) = 1/3$.

and also $P(B_i/A_k) = 0.1$, $k \neq i$ and $P(B_i/A_i) = 0.8$, $i = 1,2,3$

1.4-12 A pharmaceutical Product Consists of 100 pills in a bottle. Two Production lines used to Produce the Product are Selected with Probabilities 0.45 (line 1) and 0.55 (line 2). Each line can overfill or underfill bottles by at most 2 pills. Given that line 1 is observed, the probabilities are 0.02, 0.06, 0.88, 0.03, and 0.01 that the numbers of pills in a bottle will be 102, 101, 100, 99, and 98 respectively. For line 2, the Similar respective probabilities are 0.03, 0.08, 0.83, 0.04 and 0.02.

(a) Find the Probability that a bottle of the product will Contain 102 pills. Repeat for 101, 100, 99, and 98 pills.

(b) Given that a bottle Contains the Correct number of pills, what is the Probability it came from line 1?

(c) what is the probability that a purchaser of the product will receive less than 100 pills?

Sol:

	No. of pills	Line L_1	L_2
A_1	102	0.02	0.03
A_2	101	0.06	0.08
A_3	100	0.88	0.83
A_4	99	0.03	0.04
A_5	98	0.01	0.02

$P(L_1) = 0.95$, $P(L_2) = 0.55$

$P(102/L_1) = 0.02$ $\quad$ $P(102/L_2) = 0.03$

$P(101/L_1) = 0.06$ $\quad$ $P(101/L_2) = 0.08$

$P(100/L_1) = 0.88$ $\quad$ $P(100/L_2) = 0.83$

$P(99/L_1) = 0.03$ $\quad$ $P(99/L_2) = 0.04$

$P(98/L_1) = 0.01$ $\quad$ $P(98/L_2) = 0.02$

(a) $P(102) = P(102/L_1)\cdot P(L_1) + P(102/L_2)\cdot P(L_2) = 0.02(0.45) + 0.03(0.55)$
$= 0.0255$

lly $P(101) = 0.0710$, $P(100) = 0.8525$, $P(99) = 0.0355$
$P(98) = 0.0155$.

(b) $P(L_1/100) = \dfrac{P(100/L_1)\cdot P(L_1)}{P(100)} = \dfrac{0.88(0.45)}{0.8525} \approx 0.4645$

(c) $P(\text{pills} < 100) = P(98) + P(99) = 0.0355 + 0.0155 = 0.0510$

1.4-13 A manufacturing plant makes radios that each contains an integrated circuit (IC) supplied by three sources A, B and C. The probability that the IC in a radio came from one of the sources is 1/3, the same for all sources. ICs are known to be defective with probabilities 0.001, 0.003 and 0.002 for sources A, B and C, respectively.

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(a) What is the Probability any given radio will Contain a defective IC?
(b) If a radio Contains a defective IC, find the probability it came from Source A. Repeat for Source B and C.

Sol: Define $A = \{IC \text{ from Source } A\}$, $B = \{IC \text{ from Source } B\}$
$C = \{IC \text{ from Source } C\}$ & $D = \{\text{Defective } IC\}$

Then

$$P(D/A) = 0.001, \quad P(D/B) = 0.003, \quad P(D/C) = 0.002$$
$$P(A) = 1/3, \quad P(B) = 1/3, \quad P(C) = 1/3$$

(a) $P(D) = P(D/A)\,P(A) + P(D/B)\,P(B) + P(D/C)\,P(C)$
$= 0.001 \cdot 1/3 + 0.003 \cdot 1/3 + 0.002 \cdot 1/3 = 0.006/3 = 0.002$

(b) $P(A/D) = P(D/A) \cdot P(A) / P(D) = 0.001 \cdot 1/3 / 0.002 = 1/6.$

Similarly

$$P(B/D) = 1/2 \quad ; \quad P(C/D) = 1/3$$

1.4-14 There are three special decks of cards. The first, deck D_1, has all 52 cards of a regular deck. The Second, D_2 has only the 16 face cards of a regular deck (only 4 each of jacks, queens, kings and aces). The third D_3, has only the 36 numbered cards of a regular deck (4 twos through 4 tens). A random experiment Consists of first randomly choosing one of the three decks, then second, randomly choosing a card from the choosen deck. If $P(D_1) = 1/2$, $P(D_2) = 1/3$ and $P(D_3) = 1/6$, find the Probabilities
(a) of drawing an ace (b) of drawing a three and (c) of drawing a red card.

Sol: (a) $P(ace) = P(A/D_1) \cdot P(D_1) + P(A/D_2) \cdot P(D_2) + P(A/D_3) \cdot P(D_3)$
$= 4/52 \cdot 1/2 + 1/3 + 0 \cdot 1/6 = 19/156$

(b) $P(3) = P(3/D_1) \cdot P(D_1) + P(3/D_2) \cdot P(D_2) + P(3/D_3) \cdot P(D_3)$
$= 4/52 \cdot 1/2 + 0 \cdot 1/3 + 4/36 \cdot 1/6 = 20/351 \simeq 0.0570$

(c) Ily $P(\text{red card}) = 26/52 \cdot 1/2 + 8/16 \cdot 1/3 + 18/36 \cdot 1/6 = 1/2 = 0.5$

1.5-1 Determine whether the three events A, B, and C of example 1.4-1 are Statistically independent.

Sol: $P(A \cap B) = 28/100 = 0.28 \neq P(A) \cdot P(B) = 44/100 \cdot 62/100 = 0.273$

So A and B are not independent

$P(A \cap C) = 0.0$ while $P(A) \cdot P(C) \neq 0$ also A and C are not independent.

$P(B \cap C) = 0.24 \neq P(B) \cdot P(C) = 62/100 \cdot 32/100 = 0.198$. So B and C are not independent.

Finally $P(A \cap B \cap C) = 0.0 \neq P(A) \cdot P(B) \cdot P(C)$, So A, B and C are not statistically independent, even by pairs.

1.5-2 List the various equations that four events A_1, A_2, A_3 and A_4 must Satisfy if they are to be statistically independent.

Sol: $P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$; $P(A_1 \cap A_3) = P(A_1) \cdot P(A_3)$

$P(A_1 \cap A_4) = P(A_1) \cdot P(A_4)$; $P(A_2 \cap A_3) = P(A_2) \cdot P(A_3)$; $P(A_2 \cap A_4) = P(A_2) \cdot P(A_4)$

$P(A_3 \cap A_4) = P(A_3) \cdot P(A_4)$; $P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$.

$P(A_1 \cap A_2 \cap A_4) = P(A_1) \cdot P(A_2) \cdot P(A_4)$; $P(A_1 \cap A_3 \cap A_4) = P(A_1) \cdot P(A_3) \cdot P(A_4)$.

$P(A_2 \cap A_3 \cap A_4) = P(A_2) \cdot P(A_3) \cdot P(A_4)$ & $P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4)$.

1.5-3 Given that two events A_1 and A_2 are statistically independent, show that

(a) A_1 is independent of $\bar{A}_2$

(b) $\bar{A}_1$ is independent of A_2

(c) $\bar{A}_1$ is independent of $\bar{A}_2$

Sol: (a) $P(A_1) \cdot P(\bar{A}_2) = P(A_1) \cdot [1 - P(A_2)] = P(A_1) - P(A_1) P(A_2)$

Since A_1 and A_2 are independent this becomes

$$P(A_1) \cdot P(\bar{A}_2) = P(A_1) - P(A_1 \cap A_2) \quad \text{——— (1)}$$

$$\boxed{A_1 = (A_1 \cap \bar{A}_2) \cup (A_1 \cap A_2)}$$

So $P(A_1) = P[(A_1 \cap \bar{A}_2) \cup (A_1 \cap A_2)]$

$= P[A_1 \cap \bar{A}_2] + P[A_1 \cap A_2] - P[(A_1 \cap \bar{A}_2) \cap (A_1 \cap A_2)]$

$$\boxed{(A_1 \cap \bar{A}_2) \cap (A_1 \cap A_2) = \phi}$$

So

$$P(A_1) = P(A_1 \cap \bar{A}_2) + P(A_1 \cap A_2) \quad \text{——— (2)}$$

$$P(A_1 \cap \bar{A_2}) = P(A_1) - P(A_1 \cap A_2) \quad \text{---} \quad (2)$$

By combining (1) & (2) we have $P(A_1) \cdot P(\bar{A_2}) = P(A_1 \cap \bar{A_2})$. So A_1 and $\bar{A_2}$ are statistically independent.

(b) $P(\bar{A_1}) \cdot P(A_2) = P(A_2)[1 - P(A_1)]$

$P(\bar{A_1}) \cdot P(A_2) = P(A_2) - P(A_2) \cdot P(A_1) = P(A_2) - P(A_2 \cap A_1)$

But $A_2 = (A_2 \cap \bar{A_1}) \cup (A_2 \cap A_1)$ & $(A_2 \cap \bar{A_1}) \cap (A_2 \cap A_1) = \phi$

So $P(A_2) = P[(A_2 \cap \bar{A_1}) \cup (A_2 \cap A_1)] = P(A_2 \cap \bar{A_1}) + P(A_2 \cap A_1) - P[(A_2 \cap \bar{A_1}) \cap (A_2 \cap A_1)]$

$$P(A_2) = P(A_2 \cap \bar{A_1}) + P(A_2 \cap A_1)$$

$$P(A_2 \cap \bar{A_1}) = P(A_2) - P(A_2 \cap A_1)$$

So $\bar{A_1}$ and A_2 are statistically independent.

(c) Again repeat the procedure of (a) and (b)

$P(\bar{A_1}) P(\bar{A_2}) = [1 - P(A_1)][1 - P(A_2)] = 1 - P(A_1) - P(A_2) + P(A_1) \cdot P(A_2)$

$= 1 - P(A_1) - P(A_2) + P(A_1 \cap A_2)$

$= 1 - [P(A_1) + P(A_2) - P(A_1 \cap A_2)]$

$$P(\bar{A_1}) \cdot P(\bar{A_2}) = 1 - P(A_1 \cup A_2)$$

But since $1 - P(A_1 \cup A_2) = P(\overline{A_1 \cup A_2})$

$$\therefore \quad P(\bar{A_1}) P(\bar{A_2}) = P(\overline{A_1 \cup A_2})$$

However $\overline{A_1 \cup A_2} = \bar{A_1} \cap \bar{A_2}$. So

$$P(\bar{A_1}) \cdot P(\bar{A_2}) = P(\bar{A_1} \cap \bar{A_2})$$

$\bar{A_1}$ and $\bar{A_2}$ to be statistically independent.

1.5-4 Show that there are $2^N - N - 1$ equations required in 1.5-6.
(Hint: Recall that the binomial coefficient is the number of combinations of N things taken 'n' at a time).

Sol: Combination of 2 things taken from N things $= \binom{N}{2}$

Combination of 3 things taken from N things $= \binom{N}{3}$

$\vdots$

Combination of N things taken from N things $= \binom{N}{N}$

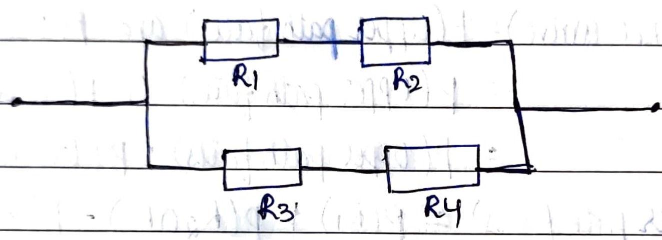
__/__/____

$$\text{Total} = \text{Sum} = \sum_{i=2}^{N} \binom{N}{i} = -\binom{N}{0} - \binom{N}{1} + \sum_{i=0}^{N} \binom{N}{i}$$

$$\text{Total} = -1 - N + \sum_{i=0}^{N} \binom{N}{i} = 2^N - N - 1$$

1.5-5 In a Communication System the Signal Sent from Point a to Point b arrives by two paths in parallel. Over each path the signal passes through two repeaters (in series). Each repeater in one path has a probability of failing (becoming an open circuit) of 0.005. This Probability is 0.008 for each repeater on the other path. All the repeaters fail independently of each other. Find the Probability that the Signal will not arrive at point b.

Sol:



$R_i = \{\text{relay } R_i \text{ fails};\ i = 1,2,3,4\} = \{R_i \text{ Opens}\}$

$$P(R_1) = P(R_2) = 0.005, \quad P(R_3) = P(R_4) = 0.008.$$

$P(\text{Signal doesnot arrive}) = P\{(R_1 \text{ or } R_2 \text{ opens}) \text{ and } (R_3 \text{ or } R_4 \text{ Opens})\}$

$= P\{(R_1 \cup R_2) \cap (R_3 \cup R_4)\}$

$= P\{R_1 \cup R_2\} \cdot P(R_3 \cup R_4)$

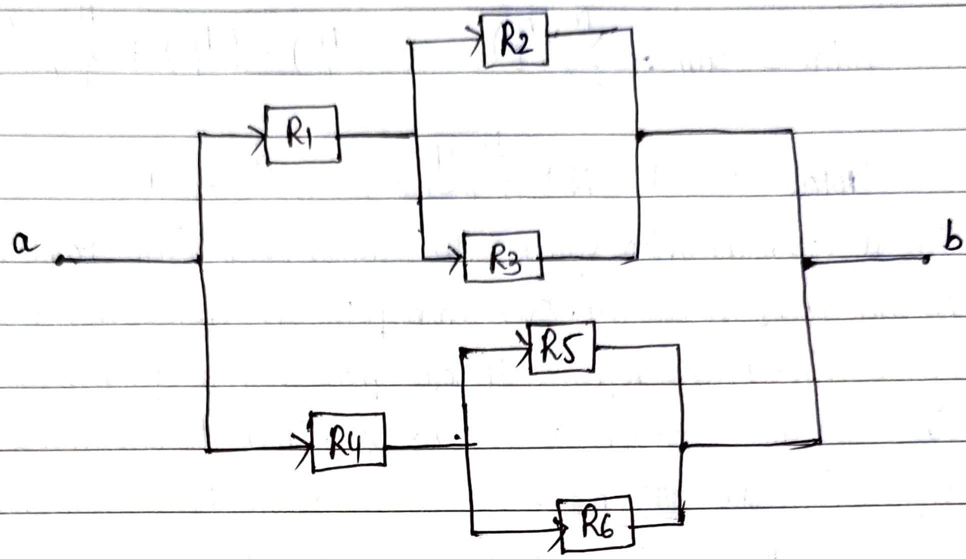
$= [P(R_1) + P(R_2) - P(R_1 \cap R_2)] \cdot [P(R_3) + P(R_4) - P(R_3 \cap R_4)]$

(Independent failures) $= [0.01 - 25(10^{-6})]\,[0.016 - 64(10^{-6})]$

$\simeq 0.00016$

1.5-6 Work Problem 1.5-5, except assume the paths and repeaters of below figure, where the probabilities of repeaters failing (independently) are $P_1 = P(R_1) = 0.005$, $P_2 = P(R_2) = P(R_3) = P(R_4) = 0.01$ and $P_3 = P(R_5) = P(R_6) = 0.05$.

__/__/__



Sol: P(signal doesnot arrive) = P(upper path fails) and P(lower path fails)

= P(upper path fails) ∩ P(lower path fails)

= P(upper path fails) · P(lower path fails)

But, P(upper path fails) = $P(R_1) + P(R_2 \cap R_3) - P(R_1 \cap R_2 \cap R_3)$

$= P_1 + P_2^2 - P_1 \cdot P_2^2 = 5.0995 \times 10^{-3}$ and

P(lower path fails) = $P_2 + P_3^2 - P_2 \cdot P_3^2 = 12.475 \times 10^{-3}$

Thus

P(signal not arrive) = $(5.0995 \times 10^{-3}) \cdot (12.475 \times 10^{-3})$

$= 63.6163 \times 10^{-6}$

1.6-1 An random experiment consists of randomly selecting one of five cities on Florida's west coast for vacation. Another experiment consists of selecting at random one of four acceptable hotels in which to stay. Define sample space S_1 and S_2 for the two experiments and a combined space $S = S_1 \times S_2$ for the combined experiment having the two subexperiments.

Sol: Let c_1, c_2, c_3, c_4 and c_5 represents the five cities and m_1, m_2, m_3 and m_4 represents the motels.

Then $S_1 = \{c_1, c_2, c_3, c_4, c_5\}$

$S_2 = \{m_1, m_2, m_3, m_4\}$. The combined sample space becomes $S = S_1 \times S_2 = \{(c_1, m_1), (c_1, m_2), (c_1, m_3), (c_1, m_4), (c_2, m_1), (c_2, m_2), (c_2, m_3), (c_2, m_4), (c_3, m_1), (c_3, m_2), (c_3, m_3), (c_3, m_4), (c_4, m_1)\ (c_4, m_2)$

___/___/______

$(C_4, m_3)\ (C_4, m_4), (C_5, m_1)\ (C_5, m_2)\ (C_5, m_3)\ (C_5, m_4)\}$

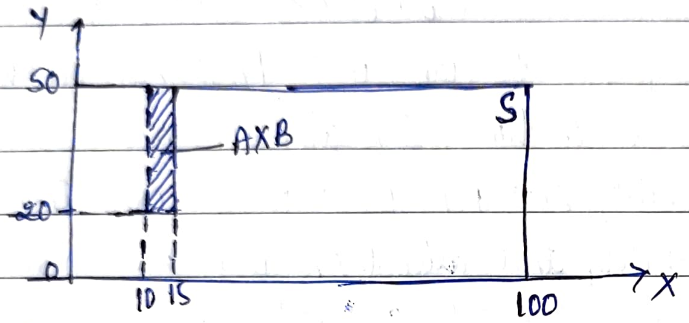
1·6-2 Sketch the area in the Combined Sample space of Example 1·6-3 Corresponding to the event $A \times B$. where

(a) $A = \{10 < x \leq 15\}$ and $B = \{20 < y \leq 50\}$

(b) $A = \{x = 40\}$ and $B = \{5 < y \leq 40\}$.

Sol:

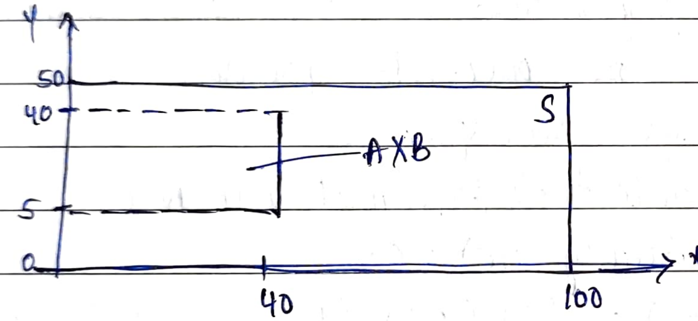
(a)



$A = \{10 < x \leq 15\}$

$B = \{20 < y \leq 50\}$

(b)



$A = \{x = 40\}$

$B = \{5 < y \leq 40\}$

1·6-3· The Six Sides of a fair die are numbered from 1 to 6. The die is rolled four times. How many Sequences of four resulting numbers are Possible?

Sol: Sequences $= 6\,(6)\,(6)\,(6) = 6^4 = 1296$

1·6-4 In a 5-card Poker game, a player is dealt 5 Cards. How many Poker hands are possible for an ordinary 52 cards deck?

Sol: Number of Poker hands $= \binom{52}{5} = \dfrac{52(51)\,50\,(49)\,(48)}{5 \times 4 \times 3 \times 2}$

$= 2,598,960.$

1·7-1 A Production line manufactures 5-gal (18·93 liter) gasoline Cans to a Volume tolerance 5%. The Probability of any one can being out of tolerance is 0·03. If four cans are Selected at random:

(a) what is the Probability they are all out of tolerance?

(b) what is the Probability of exactly two being out?

(C) what is the probability that all are in tolerance?

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Sol: This is a Bernoulli trials experiment with $N=4$, $P = p$(a can is out of tolerance) $= 0.03$

(a) $P(4 \text{ out of tolerance}) = \binom{4}{4}(0.03)^4(1-0.03)^0 = 8.1 \times 10^{-7}$

(b) $P(2 \text{ out of tolerance}) = \binom{4}{2}(0.03)^2(1-0.03)^2 \cong 5.081 \times 10^{-3}$

(c) $P(\text{all in tolerance}) = P(\text{none is out of tolerance})$

$= \binom{4}{0}(0.03)^0(1-0.03)^4 = (0.97)^4 \cong 0.8853.$

1·7-2 Space crafts are expected to land in a Prescribed recovery zone 80% of the time. Over a period of time, Six aircrafts land.

(a) Find the Probability that none lands in the prescribed zone.

(b) Find the Probability that atleast one will land in the prescribed zone.

(c) The landing program is called Successful if the Probability is 0·9 (or) more that three (or) more out of six spacecraft will land in the prescribed zone. Is the program Successful.

Sol: This is a Bernoulli trials Experiment with $N=6$,

$P = P(\text{land in recovery zone}) = 0.8$

(a) $P(\text{none in zone}) = \binom{6}{0}(0.8)^0(1-0.8)^6 = (0.2)^6 = 6.4 \times 10^{-5}$

(b) $P(\text{atleast one in zone}) = 1 - P(\text{none in zone}) = 1 - 6.4 \times 10^{-5} = 0.999936$

(c) $P(\text{Success}) = P(3 \text{ in zone}) + P(4 \text{ in zone}) + P(5 \text{ in zone}) + P(6 \text{ in zone})$

$= \binom{6}{3}(0.8)^3(0.2)^3 + \binom{6}{4}(0.8)^4(0.2)^2 + \binom{6}{5}(0.8)^5(0.2) + \binom{6}{6}(0.8)^6(0.2)^0$

$\simeq 0.983.$

Yes, the Program is successful.

1·7-3 In the Submarine Problem of Example 1·7-1, find the probabilities of sinking the carrier when fewer ($N=2$) (or) more ($N=4$) torpedoes are fired.

Sol: For $N=2$

$P\{\text{carrier sunk}\} = P\{2 \text{ hits}\} = \binom{2}{2}(0.4)^2(0.6)^0 = 0.16$

For $N=4$

$P\{\text{carrier sunk}\} = P\{2 \text{ hits}\} + P\{3 \text{ hits}\} + P\{4 \text{ hits}\}$

$= \binom{4}{2}(0.4)^2(0.6)^2 + \binom{4}{3}(0.4)^3(0.6) + \binom{4}{4}(0.4)^4(0.6)^0$

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1·7-4 A student is known to arrive late for class 40% of the time. If the class meets five times each week find:

(a) the probability the student is late for atleast three classes in a given week, &

(b) the probability the student will not be late at all during given week.

Sol: $N = 5, \quad P = 0.4$

(a) p(late 3 times or more in one week) =

$$= \binom{5}{3}(0.4)^3(0.6)^2 + \binom{5}{4}(0.4)^4(0.6) + \binom{5}{5}(0.4)^5(0.6)^0$$

$$= 0.2304 + 0.0768 + 0.01024$$

$$= 0.31744$$

(b) P(not late at all) $= \binom{5}{0}(0.4)^0(0.6)^5 = 0.07776$.

1·7-5 An airline in a small city has five departures each day. It is known that any given flight has a probability of 0·3 of departing late. For any given day find the probabilities that:

(a) no flights depart late $\qquad N = 5, \; P = 0.3$

(b) all flights depart late, and (c) three or more depart on time.

Sol: (a) $P(\text{no late}) = \binom{5}{0}(0.3)^0(0.7)^5 = 0.16807$

(b) $P(\text{all late}) = \binom{5}{5}(0.3)^5(0.7)^0 = 0.00243$

(c) $P(3 \text{ or more on time}) = P(0 \text{ late}) + P(1 \text{ late}) + P(2 \text{ late})$

$$= 0.16807 + \binom{5}{1}(0.3)^1(0.7)^4 + \binom{5}{2}(0.3)^2(0.7)^3$$

$$= 0.83692.$$

1·7-6 The local manager of the airline of problem 1·7-5 desire to make sure that the probability that all flights leave on time is 0·9. What is the largest probability of being late that the individual flights can have if the goal is to be achieved? Will the operation have to be improved significantly?

Sol: $P(\text{all on time}) = P(0 \text{ late}) = \binom{5}{0} x^0 (1-x)^5 = (1-x)^5 = 0.9$.

Thus $x = P(\text{a flight is late}) = 1 - 0.9^{1/5} = 0.020852$.

Yes — to reduce 0·3 to 0·020852.

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1.7-7 A man wins in a gambling game if he gets two heads in five flips of a biased coin. The probability of getting a head with the coin is 0.7.
(a) find the probability the man will win. should he play this game?
(b) what is his Probability of winning if he wins by getting at least four heads in five flips? $N=5,\ P=0.7$

Sol: (a) $P(\text{win}) = P(2\text{ heads}) = \binom{5}{2}(0.7)^2(0.3)^3 = 0.13230$. NO

(b) $P(\text{win}) = P(4\text{ heads}) + P(5\text{ heads})$

$= \binom{5}{4}(0.7)^4(0.3)^1 + \binom{5}{5}(0.7)^5(0.3)^0 = 0.36015 + 0.16807$

$= 0.52822$.

1.7-8 A rifleman can achieve a "marksman" award if he passes a test. He is allowed to fire six shots at a target's bull's eye. If he hits the bull's eye with atleast five of his six shots he wins a set. He becomes a marksman only if he can repeat the feat three time striaght, that is, if he can win three striaght sets. If his probability is 0.8 of hitting a bull's eye on any one shot, find the probabilities of his: $N=6,\ P=0.8$
(a) winning a set and (b) becoming a marksman.

Sol: (a) $P(\text{win set}) = P(5\text{ hits in }6) + P(6\text{ hits in }6)$

$= \binom{6}{5}(0.8)^5(0.2)^1 + \binom{6}{6}(0.8)^6(0.2)^0$

$= 0.393216 + 0.262144 = 0.65536$.

(b) Bernoulli's trial

$P(3\text{ sets of }3) = \binom{3}{3}(0.65536)^3(1-0.65536)^0$

$= 0.28147$

1.7-9 A ship can succesfully arrive at its destination if its engine and its satellite navigation system donot fail enroute. If the engine and navigation system are known to fail independently with respective Probabilities of 0.05 and 0.001, what is the Probability of a successful arrive?

Sol: $P(\text{Successful arrival}) = P\{\text{engine Survives and navigation System Survives}\} = P\{\text{engine Survives}\} \cdot P\{\text{system navigation system survives}\}$

$= [1-P(\text{engine fails})][1-P(\text{navigation System fails})] = 0.95(0.999)$

$= 0.94905$

1.7-10 At a certain military installation 6 similar radars are placed in operation. It is known that a radar's probability of failing to operate before 500 hours of "on" time have accumulated is 0.06. What are the probabilities that before 500 hours have elapsed (a) all will operate (b) all will fail and (c) only one will fail.

Sol: (a) $P(\text{all operates}) = P(0\text{ fail}) = \binom{6}{0}(0.06)^0(0.94)^6 = 0.6899$.

(b) $P(\text{all fail}) = \binom{6}{6}(0.06)^6(0.94)^0 = 4.666(10^{-8})$

(c) $P(1\text{ fails}) = \binom{6}{1}(0.06)^1(0.94)^5 = 0.2642$.

1.7-11 A particular model of automobile is recalled to fix a mechanical problem. The ~~problem~~ probability that a car will be properly repaired is 0.9. During the week a dealer has eight cars to repair.

(a) What is the probability that two or more of the eight cars will have to be repaired more than once?

(b) What is the probability all eight cars will be properly repaired?

Sol: (a) $P(2\text{ or more not repaired}) = 1 - P(0\text{ not repaired}) - P(1\text{ not repaired})$

$$= 1 - \binom{8}{0}(0.1)^0(0.9)^8 - \binom{8}{1}(0.1)^1(0.9)^7$$

$$= 1 - 0.4305 - 0.3826$$

$$= 0.1869$$

(b) $P(8\text{ properly repaired}) = P(0\text{ not repaired}) \cong 0.4305$.

1.7-12 In a large hotel it is known that 99% of all guests return room keys when checking out. If 250 engineers check out after a large conference, what is the probability that not more than three will fail to return their keys? [Hint: use the approximation of (1.7-8)].

Sol: $P(0\text{ fail to return key}) = (NP)^{k=0}e^{-NP}/(k=0)! = (2.5)^0 e^{-2.5}/0! = 0.0821$

$P(1\text{ fail to return key}) = (2.5)^1 e^{-2.5}/1! = 0.2052$

$P(2\text{ fail to return key}) = (2.5)^2 e^{-2.5}/2! = 0.2565$

$P(3\text{ fail to return key}) = (2.5)^3 e^{-2.5}/3! = 0.2138$

$\therefore P(\text{no more than 3 fail to return keys}) = 0.2052 + 0.0821 + 0.2565 + 0.2138 = 0.7576$